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SOLUTION OF THE TWO-DIMENSIONAL NEUTRON TRANSPORT EQUATION BY A HIERARCHICAL FINITE ELEMENT METHOD

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ABSTRACT

Following the results obtained by Marchuk *et al.* concerning the variational principles for kinetic equations, a hierarchical finite element method is proposed for solving the two-dimensional neutron transport equation. The combination of mesh refinement with additional basis functions leads to an accurate and efficient iterative solution of the resulting block linear systems. Numerical results are provided to illustrate the efficiency of the method.

KEY WORDS Neutron transport equation SOR methods Finite element method

INTRODUCTION

The hierarchical finite element method (HFEM), proposed in the early 1970s, is now widely used in engineering computations[1–5]. It has been shown that as a tool for computation, the HFEM improves the accuracy of the finite element mesh[4]. The specificity of this method lies in the adaptive choice of the interpolation space by successive approximations. It has also been shown that for the p version of the finite element method a stable process is obtained for hierarchical basis functions that are normalized and almost orthogonal in energy[7–12].

In this paper we propose a HFEM for the neutron transport equation which has numerous applications in nuclear engineering, meterology, biology, etc. Numerical problems involving the neutron transport equation are generally arduous because of their multi-dimensionality, the lack of smoothness of the solutions, asymmetry of operators and a number of other singularities. The intensive development of transport computations began in the early 1950s and now constitutes an independent branch of applied mathematics[13–16]. When applied to the neutron transport problem, the HFEM produces for every new hierarchical space V_i a matrix with desirable properties: symmetry, positive definiteness, diagonal dominance, and block-structure. This permits use of a large spectrum of efficient numerical methods to minimize the cost of processing, which escalates rapidly with the introduction of new hierarchical spaces if appropriate techniques are not used. We focus on the stationary two-dimensional neutron transport equation. From the results obtained by Marchuk *et al.*[17] we propose an application of the HFEM based on the Ritz variational method. This results in a familiar block system of linear algebraic equations which are solved iteratively, using a SOR method with relaxation parameter optimization. Numerical results focusing mainly on the methods rather than on the physics of the problem are provided to illustrate the advantages of the HFEM.

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VARIATIONAL FORMULATION

Consider the following boundary value problem for the neutron transport equation[13],

$$\begin{cases} (\vec{\Omega}.\vec{\nabla})\varphi + \sigma\varphi = \dfrac{G_s}{2\pi}\displaystyle\int_0^{2\pi} \varphi(x_1, x_2, \theta')\,\mathrm{d}\theta' + f \\ \varphi = \rho_r(x_1, x_2, \theta); \qquad (x_1, x_2) \in \partial D- \end{cases} \tag{1}$$

where,

D is a convex bounded domain of $\mathbb{R}^2$ with a smooth boundary, ∂D

$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ is the unit vector of the outer normal to ∂D

$\vec{\Omega} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} (\theta \in [0, 2\pi])$ is a unit vector that assumes values on a unit circle centred at the origin

Let $\Omega =]0, 2\pi[$ and:

$$\partial D- = \{(x_1, x_2) \in \partial D / \vec{n}.\vec{\Omega} \leqslant 0\}$$
$$\partial D+ = \{(x_1, x_2) \in \partial D / \vec{n}.\vec{\Omega} \geqslant 0\}$$

We introduce,

$$(\vec{\Omega}.\vec{\nabla})\varphi = \cos\theta \frac{\partial \varphi}{\partial x_1} + \sin\theta \frac{\partial \varphi}{\partial x_2}$$

$$s_0\varphi = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x_1, x_2, \theta')\,\mathrm{d}\theta'$$

Furthermore, we assume that: $f, \varphi \in L(D \times \Omega)$ and $\sigma, \sigma_s \in L_2(D)$. As proposed by Marchuk *et al.*[17] we introduce a function space,

$$W_1 = \left\{ u : (u, \omega)_{W_1} = \int_D \mathrm{d}x \int_0^{2\pi} \mathrm{d}\theta \left(\frac{1}{\sigma}(\vec{\Omega}.\vec{\nabla})u.(\vec{\Omega})\omega + \sigma u\omega \right) + \right.$$
$$\left. \int_0^{2\pi} \mathrm{d}\theta \int_{\partial D} \mathrm{d}x |\vec{\Omega}\vec{n}| u\omega; \qquad \|u\|_{W_1} = (u, u)_{W_1}^{1/2} < \infty \right\}$$

In the Hilbert space W_1 a variational Ritz formulation is obtained after rendering symmetric the operator in (1)[13]. The energy function $J(\varphi)$ is given by[17],

$$J(\varphi) = [v, v] - 2(f, v) - 2(Bf, (\vec{\Omega}.\vec{\nabla})v) - 4\int_0^{2\pi} \mathrm{d}\theta \int_{\partial D-} \mathrm{d}x |\vec{\Omega}\vec{n}|.\varphi_\Gamma.v \tag{2}$$

where,

$$[v, v] = \int_{\partial D \times \Omega} \mathrm{d}x\,\mathrm{d}\theta |\vec{\Omega}\vec{n}| u.\omega + \int_{\partial D \times \Omega} \mathrm{d}x\,\mathrm{d}\theta\, B(\vec{\Omega}.\vec{\nabla})\omega +$$
$$\int_{\partial D \times \Omega} \mathrm{d}x\,\mathrm{d}\theta(\sigma u\omega - (\sigma_s S_0 u).\omega) \tag{3}$$

and B is a linear operator defined as follows,

$$B\omega = \frac{\omega}{\sigma} - + \frac{\sigma_s}{\sigma(\tau - \sigma_s)} S_0 \omega$$

Let,

$$L(v) = (f, v) + (Bf, (\vec{\Omega} . \vec{\nabla})v) + 2 \int_{\Omega \times \partial D -} \mathrm{d}x \, \mathrm{d}\theta |\vec{\Omega} . \vec{n}| \varphi_\Gamma . v \tag{4}$$

Then, the boundary value problem (1) is equivalent to the following extremal problem,

$$J(\varphi) = \min_{v \in W_1} [v, v] - 2L(v) \tag{5}$$

HIERARCHICAL FINITE ELEMENT FORMULATION

Basic equations

Let,

$$\Phi_x = (\Phi_x^1 \ldots \Phi_x^j \ldots \Phi_x^M)$$

be a system of spatial finite element basis functions.

P_1 denotes the set of two-variable polynomials of degree one, and Φ_x^j is an element of P_1. On a nodal point a_i we have,

$$\Phi_x^j(a_i) = \delta_{ij} \tag{6}$$

where δ_{ij} is the Kroneker delta. The φ_{ij} are the nodal variables. Let,

$$\tilde{\varphi}_i = \begin{pmatrix} \varphi_{i1} \\ \vdots \\ \varphi_{ij} \\ \vdots \\ \varphi_{iM} \end{pmatrix} \tag{7}$$

The flux $\varphi(x_1, x_2, \theta)$ may be approximated by the product,

$$\hat{\varphi}(x_1, x_2, \theta) = \Phi_x . \sum_{i=1}^{N} \Phi_i(\theta) \tilde{\varphi}_i \tag{8}$$

$\phi_i(\theta)$, $1 \leqslant i \leqslant N$ are the basic functions in angular variable.

$\hat{\varphi}(x_1, x_2, \theta)$ is a solution of (5) if and only if:

$$\sum_{i=1}^{N} \tilde{\varphi}_i [\Phi_i(\theta)\Phi_x, \Phi_l(\theta)\Phi_x] = L(\Phi_l(\theta)\Phi_x) = \tilde{b}_1 \qquad 1 \leqslant l \leqslant N \tag{9}$$

The system (9) may then be written as:

$$\sum_{i=1}^{N} A_{i,l} \tilde{\varphi}_i = \tilde{b}_1 \tag{10}$$

where the $A_{i,l}$ ($1 \leqslant i \leqslant N$, $1 \leqslant l \leqslant M$) are $m \times m$ matrices.

Theorem 1. The following properties are characteristic of the system stiffness matrix defined in (10):

(a) $$A_{i,l} = A_{1,l} \tag{11}$$

(b) $$A_{il} = A_{il}^t$$

(c) if the $\Phi_l(\theta)$ are three-nodal approximation functions then system (10) is block-tridiagonal,
(d) the matrices A_{il} are positive definite,
(e) all the matrice A_{il} are invertible.

Proof. Properties (a), (b), (d) and (e) follow from the fact that $[.,.]$ in (a) is a scalar product. To prove (c) note that if the $\Phi_l(\theta)$ are three-nodal approximation functions, then the supports $\mathbb{K}_1$ of the functions Φ_l are such that,

$$\mathbb{K}_i \cap \mathbb{K}_j \text{ is empty if } |i-j| \geqslant 2$$

In this case the sets $(\mathbb{K}_i \cap \mathbb{K}_j) \times D$ and $(\mathbb{K}_i \cap \mathbb{K}_j) \times \partial D$ are empty.

Thus,

$$[\Phi_i(\theta)\Phi_x^j, \Phi_l(\theta)\Phi_x^m] = 0 \tag{12}$$

$$\forall j, m \in \{1, 2, 3, \ldots, M\}; \qquad |i-j| \geqslant 2$$

Therefore, for a given i only the following matrices are not nul:

$$A_{i-1i}, A_{ii}, A_{i+1i} \qquad (i = 2, \ldots, N-1)$$

Thus the system (10) is block-tridiagonal.

Hierarchical formulation for the angular variable

We approximate the function $\tilde{\varphi}(\theta)$ in the interval $[0, 2\pi]$ with,

$$\tilde{\varphi}(\theta) = \sum_{i=1}^{N} \Phi_i(\theta)\tilde{\varphi}_l$$

$$\tilde{\varphi}(\theta) = \sum_{i=1}^{N} \Phi_{ib}\tilde{\varphi}_{ib} + \sum_{i=N+1}^{S} \Phi_{ih}\tilde{\varphi}_{ih} \tag{13}$$

where Φ_{ib} denotes the three-nodal reference element basic functions and the Φ_{ih} are the hierarchical basic functions. Let:

$$\Phi_1(\varepsilon) = 1 - \varepsilon \text{ on the reference element } (0, 1)$$

$$\Phi_2(\varepsilon) = 1 - |\varepsilon| \text{ on the reference element } (-1, 0, 1)$$

$$\Phi_3(\varepsilon) = \varepsilon \text{ on the reference element } (0, 1)$$

Furthermore, we let

$$\Phi_b = (\Phi_1, \Phi_2, \Phi_3)$$

be the set of reference functions from which the basic functions are constructed. Now consider new reference functions Φ_4, Φ_5 defined on the reference element $(-1, 0, 1)$ as follows,

$$\Phi_4(\varepsilon) = 1 - \varepsilon^2, \Phi_5(\varepsilon) = \cos\frac{\pi\varepsilon}{2}$$

Let

$$\Phi_h = (\Phi_4, \Phi_5)$$

be the class of reference functions from which the hierarchical space of order i is constructed. Then from (13), system (10) may be partitioned as follows,

$$\sum_{i=1}^{N} A_{i,l}\tilde{\varphi}_i + \sum_{i=N+1}^{S} A_{i,l}\tilde{\varphi}_i = \tilde{b}_l \qquad 1 \leqslant l \leqslant s \tag{14}$$

or in matrix form,

$$A_h\Phi = \begin{bmatrix} A_{bb} & A_{bh} \\ A_{hb} & A_{hh} \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_b \\ \tilde{\Phi}_h \end{bmatrix} = \begin{bmatrix} \tilde{B}_b \\ \tilde{B}_h \end{bmatrix} \tag{15}$$

where $\tilde{\Phi}_b$ and $\tilde{\Phi}_h$ are the nodal variables relative to the basic functions and the hierarchical functions, respectively. $\tilde{B}_b$ and $\tilde{B}_h$ are the corresponding forcing vectors:

$$\tilde{\Phi}_b = \begin{pmatrix} \tilde{\varphi}_1 \\ \vdots \\ \tilde{\varphi}_i \\ \vdots \\ \tilde{\varphi}_N \end{pmatrix}; \quad \tilde{\Phi}_h = \begin{pmatrix} \tilde{\varphi}_{N+1} \\ \vdots \\ \tilde{\varphi}_i \\ \vdots \\ \tilde{\varphi}_S \end{pmatrix}; \quad \tilde{B}_b = \begin{pmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_N \end{pmatrix}; \quad \tilde{B}_b = \begin{pmatrix} \tilde{b}_{N+1} \\ \vdots \\ \tilde{b}_i \\ \vdots \\ \tilde{b}_S \end{pmatrix}$$

The matrix A_H has the following properties.

Theorem 2.

(i) A_{bb} is block-tridiagonal, symmetric and positive definite
(ii) $A_{bh} = A_{hb}^t$
(iii) A_{hh} is block-tridiagonal, symmetric and positive definite.

These properties follow from the properties of the scalar product [.,.]. Therefore matrix A_H is block-tridiagonal, positive definite and has diagonal dominance. This makes it particularly attractive to use iterative methods for solving system (15).

If several refinements by hierarchical functions are made then system (15) may be further partitioned. As an example for a basic discretization involving k basic functions and two refinements involving s and t hierarchical functions respectively, the resulting system is,

$$\begin{bmatrix} A_{kk} & A_{ks} & A_{kt} \\ A_{sk} & A_{ss} & A_{st} \\ A_{tk} & A_{ts} & A_{tt} \end{bmatrix} \begin{bmatrix} \tilde{\phi}_b \\ \tilde{\phi}_{n1} \\ \tilde{\phi}_{n2} \end{bmatrix} = \begin{bmatrix} \tilde{B}_b \\ \tilde{B}_{n1} \\ \tilde{B}_{n2} \end{bmatrix}$$

Determination of the elements of the matrix $A_{i,l}$

We begin by making the following definitions. Let:

Ω_i be the support of the function Φ_i;
$\Phi_{ij} = \Phi_i \cap \Phi_j$

$$b_i = \int_{\Omega_i} \Phi_i(\theta)\cos\theta \, d\theta$$

$$e_i = \int_{\Omega_i} \Phi_i(\theta) \sin\theta \, d\theta$$

$$d_i = \int_{\Omega_i} \Phi_i(\theta) \, d\theta$$

$$c_{i,l} = \int_{\Omega_{il}} \sin\theta \, \Phi_i(\theta)\Phi_l(\theta) \, d\theta$$

$$b_{i,l} = \int_{\Omega_{il}} \cos\theta \, \Phi_i(\theta)\Phi_l(\theta) \, d\theta$$

$$c_{i,l} = \int_{\Omega_{il}} \sin\theta \, \Phi_i(\theta)\Phi_l(\theta) \, d\theta$$

$$d_{i,l} = \int_{\Omega_{il}} \Phi_i(\theta)\Phi_l(\theta) \cos^2\theta \, d\theta$$

$$e_{i,l} = \int_{\Omega_{il}} \Phi_i(\theta)\Phi_l(\theta) \cos\theta \sin\theta \, d\theta$$

$$f_{i,l} = \int_{\Omega_{il}} \Phi_i(\theta)\Phi_l(\theta) \sin^2\theta \, d\theta$$

$$l_{i,l} = \int_{\Omega_{il}} \Phi_i(\theta)\Phi_l(\theta)\theta \, d\theta$$

$$B = \int_{\partial D+} dx \, n_1 \Phi_x^t \Phi_x - \int_{\partial D-} dx \, n_1 \Phi_x^t \Phi_x$$

$$C = \int_{\partial D+} dx \, n_2 \Phi_x^t \Phi_x - \int_{\partial D-} dx \, n_2 \Phi_x^t \Phi_x$$

$$D = \int_D \frac{1}{\sigma} \left(\frac{\partial \Phi_x}{\partial x_1}\right)^t \left(\frac{\partial \Phi_x}{\partial x_1}\right) dx_1 \, dx_2$$

$$E = \int_D \frac{1}{\sigma} \left[\left(\frac{\partial \Phi_x}{\partial x_1}\right)^t \left(\frac{\partial \Phi_x}{\partial x_1}\right) \left(\frac{\partial \Phi_x}{\partial x_2}\right)^t \left(\frac{\partial \Phi_x}{\partial x_1}\right) \right] dx_1 \, dx_2$$

$$F = \int_D \frac{1}{\sigma} \left(\frac{\partial \Phi_x}{\partial x_2}\right)^t \left(\frac{\partial \Phi_x}{\partial x_2}\right) dx_1 \, dx_2$$

$$G = \int_D \frac{\sigma_s}{\sigma(\sigma - \sigma_s)} \left(\frac{\partial \Phi_x}{\partial x_1}\right)^t \left(\frac{\partial \Phi_x}{\partial x_1}\right) dx_1 \, dx_2$$

$$H = \int_D \frac{\sigma_s}{\sigma(\sigma - \sigma_s)} \left(\frac{\partial \Phi_x}{\partial x_1}\right)^t \left(\frac{\partial \Phi_x}{\partial x_2}\right) dx_1 \, dx_2$$

$$I = \int_D \frac{\sigma_s}{\sigma(\sigma - \sigma_s)} \left(\frac{\partial \Phi_x}{\partial x_2}\right)^t \frac{\partial \Phi_x}{\partial x_1} dx_1 \, dx_2$$

$$K = \int_D \frac{\sigma_s}{\sigma(\sigma - \sigma_s)} \left(\frac{\partial \Phi_x}{\partial x_2}\right)^t \frac{\partial \Phi_x}{\partial x_2} dx_1 \, dx_2$$

$$L = \int_D \sigma \Phi_x^t \Phi_x \, dx_1 \, dx_2$$

$$M = \int_D \sigma_s \Phi_x^t \Phi_x \, dx_1 \, dx_2$$

It can be verified that,

$$A_{i,l} = b_{i,l}B - c_{i,(}C + d_{i,l}D + e_{i,l}E + f_{i,l}F + \frac{1}{2\pi}(b_i b_l G + c_i c_l H + b_i c_l I + c_i c_l k) + l_{i,l}L - \frac{1}{2\pi} d_i d_l M \tag{16}$$

$$i, 1 \in \{1, \ldots, N\}$$

NUMERICAL SOLUTION OF HIERARCHICAL SYSTEMS

In a sequence of m hierarchical spaces, we have to solve $(m + 1)$ systems of equations of the type shown in (15). However, we are interested only in the solution to the last system. By adaptively constructing the final interpolation space, we find approximate solutions to the first m systems. This solution process may be costly if appropriate powerful algorithms are not involved. The system corresponding to the basic approximation in the hierarchical space V_0 is:

$$A_{bb}\tilde{\Phi}_b = \tilde{B}_b \tag{17}$$

Let $\tilde{\phi}_b$ be the solution of (17) obtained by the cyclic reduction method[19]. In the ith hierarchical space V_i $(i > 0)$, the equations is partitioned as follows,

$$\begin{bmatrix} A_{bb} & A_{bh} \\ A_{hb} & A_{hh} \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_b \\ \tilde{\Phi}_h \end{bmatrix} = \begin{bmatrix} \tilde{B}_b \\ \tilde{B}_h \end{bmatrix} \tag{18}$$

Let $\tilde{\Phi}_h^0$ be an initial approximation for $\tilde{\Phi}_h$ obtained from the system,

$$\begin{bmatrix} A_{bb} & 0_{bh} \\ A_{hb} & D_{hh} \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_b \\ \tilde{\Phi}_h \end{bmatrix} = \begin{bmatrix} \tilde{B}_b \\ \tilde{B}_h \end{bmatrix} \tag{19}$$

where D_{hh} is a matrix consisting of diagonal blocks of A_{hh}, with 0_{bh} the null matrix. We then have the following approximate initial solution for the system (18):

$$\tilde{\Phi}^0 = \begin{bmatrix} \tilde{\Phi}_b^0 \\ \tilde{\Phi}_h^0 \end{bmatrix}; \quad \tilde{\Phi}_h^0 = \begin{bmatrix} \tilde{\varphi}_{N+1}^0 \\ \tilde{\varphi}_{N+k}^0 \\ \tilde{\varphi}_s^0 \end{bmatrix} \tag{20}$$

where,

$$\tilde{\varphi}_{N+k}^0 = D_{N+k,N+k}^{-1} \cdot \left(\tilde{b}_{N+k} - \sum_{i=N+1}^{S} A_{i,N+k} \tilde{\varphi}_i^0 \right); \qquad k = 1, \ldots, S - N$$

It is obvious that the solution $\tilde{\Phi}_b$ in (17) will, in general, differ from $\tilde{\Phi}_b$ in (18) because of the connection matrix A_{bh}. But if (17) is solved and an appropriate number of hierarchical functions

are included in the finite element discretization, then we expect that the available solution will be a good approximation for $\tilde{\Phi}$ in (18). We now apply a SOR algorithm to the system (18), formally written as:

$$\begin{cases} A_{bb}\tilde{\Phi}_b^{(i+1)} = \tilde{B}_b - A_{bh}\tilde{\Phi}_h^{(i)} = F_i & (21) \\ A_{hh}\tilde{\Phi}_h^{(i+1)} = \tilde{B}_b - A_{hb}\tilde{\Phi}_h^{(i)} = G_i & (22) \end{cases}$$

Let,

$$U = (U_1, U_2, \ldots, U_N)^t \text{ be the exact solution to (21)}$$

and $U^{(n)}$ its approximation at iteration n. The relation for the SOR method is the following:

$$A_{ii}U_i^{(n+1)} = w\left(-\sum_{l=1}^{i-1} A_{il}U_l^{(n+1)} - \sum_{l=i+1}^{N} A_{il}U_l^{(n)} + F_i \right) + (1-w)A_{ii}U_i^{(n)} \qquad (23)$$

$$1 \leqslant i \leqslant N$$

Now let,

$$\partial^{(n)} = U^{(n)} - U^{(n+1)}$$

be the difference vector, w the relaxation parameter, I the unity matrix and

$$L_w = (I - wL)^{-1}(wU + (1-w)I)$$

the iteration matrix. The method converges if and only if $0 < w < 2$. The problem of the rate of convergence is posed in terms of determining an optimal parameter $\tilde{w}$ such that

$$\rho(L_{\tilde{w}}) = \min_{0<w<2} \rho(L_w)$$

where $\rho(T)$ is the spectral radius of T. We define a vector norm

$$\|u\|_2 = \left(\sum_{i=1}^{N} u_i^2 \right)^{1/2}$$

The procedure used involves three steps:

(a) Computation of a new estimation of $\tilde{w}$

Suppose the Jacobi method is used for system (21). Then the iteration matrix is $B = D^{-1}(L+U)$ and its eigenvalues μ_i are real and such that[18]:

$$\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_N \text{ with } \mu_1 \geqslant 0 \text{ and } \mu_N \leqslant 0$$

Let w_1 be the current estimation of $\tilde{w}$. A new approximation ω of $\tilde{\omega}$ is computed from:

$$w = 2/(1 + \sqrt{1 - (\tilde{\mu}_1)^2}) \qquad (24)$$

where $\tilde{\mu}_1$ is an approximation of μ_1 given by:

$$\tilde{\mu}_1 = (R^{(n)} + w_1 - 1)/(w_1(R^{(n)})^{1/2}) \qquad (25)$$

Here,

$$R^{(n)} = \frac{\|\partial^{(n)}\|_2}{\|\partial^{(n-1)}\|_2}$$

(b) Validation of new estimation of $\tilde{w}$

Let w be a new approximation of $\tilde{\omega}$ computed from (24). This new value is accepted if:

$$\begin{cases} \rho(L_w) \leqslant (w_1 - 1) \text{ for } w < w_1 \\ \dfrac{\|\partial^{(n)}\|_2}{\|\partial^{(n-1)}\|_2} < (w_1 - 1) \text{ for } w \geqslant w_1 \end{cases} \tag{26}$$

$$F = 0,73$$

(c) Termination criterion

The termination criterion used for stopping the iteration is the following:

(a) $w < w_1$

$$\frac{1}{1-H} \max_i \frac{\|u_i^{(n+1)} - u_i^{(n)}\|}{\|u_i^{n+1}\|} < \mu \tag{27}$$

where

$$H = \max(w - 1, R^{(n)})$$

(b) $w \geqslant w_1$

$$\frac{1}{1-H} H \leqslant \mu \tag{28}$$

Now, a new hierarchical function $\Phi_k(\theta)$ is introduced. It follows from a variation of the energy functional:

$$\Delta J_k = -2\left[\sum_{j=1}^{M} \eta_j^2\right]; \qquad k = 1, \ldots, S - N \tag{29}$$

where

$$\eta_j^2 = \frac{(2(\Phi_x^j \Phi_k(\theta)) - \sum_{i=1}^{N} A_{i,k}^{(j)} \cdot \tilde{\varphi}_i^0)^2}{a_{j,k}^{k,k}} \tag{30}$$

$A_{i,k}^{(j)}$ is the jth line of the matrix $A_{i,k}$

and

$$a_{j,j}^{k,k} = [\Phi_k(\theta)\Phi_x^j, \Phi_k(\theta)\Phi_x^j]$$

Therefore the algorithm for construction of the hierarchical spaces is as follows:

Given a function $\Phi_k(\theta)$, compute

$$\eta_k^2; \qquad k = 1, \ldots, M$$

Define

$$\eta_{\max}^2 = \max_k \eta_k^2$$

The function $\phi_k(\theta)$ is introduced if:

$$\eta_k^2 > \alpha \eta_{\max}^2 \, \forall k \in \{1, \ldots, M\}; \qquad \alpha \in [0, 1]$$

NUMERICAL RESULTS

As a simple application, we consider a boundary value problem for a stationary two-dimensional neutron transport equation in the case of isotropic scattering, with an anisotropic source and vacuum boundary conditions:

$$f(x_1, x_2, \theta) = \exp(\theta)[x_2(1-x_2)(1-2x_1)\cos\theta + x_1(1-x_1)(1-2x_2)\sin\theta + 2x_1x_2(1-x_1)(1-x_2)] - (e^{2\pi}-1)x_1x_2(1-x_1)(1-x_2)$$

$$\sigma_s = 1$$

$$\sigma = 2$$

and,

$$\begin{cases} \cos\theta\dfrac{\partial\varphi}{\partial x_1} + \sin\theta\dfrac{\partial\varphi}{\partial x_2} + \sigma\varphi = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\varphi(x_1, x_2, \theta')\,d\theta' + f(x_1, x_2, \theta) \\ \varphi = 0 \text{ sur } \partial D - \end{cases}$$

$$\varphi(x_1, x_2, \theta) \geqslant 0, x \in D$$

where,

$$D =]0,1[x]0,1[$$

We focus on the performance of various methods for solving the resulting block linear systems, rather than on the physical aspects of the problem. Several iterative algorithms were used to solve the test problem, with a view to evaluating their efficiency. These are: the Jacobi, the Gauss–Seidel (SEIDEL), the adaptive Jacobi (JA), the SOR and the SOR with adaptation of the relaxation parameter (SORA) methods.

The results clearly show the advantage of the HFEM discretization as compared to a non-hierarchical finite element approach (FEM). *Table 1* shows that the SORA method leads to the fastest rate of convergence of the radio $R^{(n)} = \partial^{(n)}/\partial^{(n-1)}$ towards λ_1, the spectral radius of the iteration matrix. This is due to the good initial approximations obtained from (17) and (19), and the minimal effect on the solution to the problem, of introducing new hierarchical functions beyond a certain point.

Table 2 shows similar results in the case of FEM. These results are in agreement with those presented in Hageman and Young[18], and show that the spectral radius of the iteration matrix decreases with the introduction of new hierarchical basis functions, up to a point. This in turn leads to faster convergence towards a solution, as can be seen from *Table 3*.

Table 1 Estimation of the spectral radius of the iteration matrix by different iterative methods using a HFEM

	Number of iterations		
Method	30	40	60
SOR $w = 1.9$	0.900377	0.871481	0.860081
SEIDEL	0.910471	0.887817	0.844671
Jacobi	0.930842	0.918112	0.880014
JA	0.770088	0.730181	0.710394
SORA	0.694617	0.694602	0.694602

Table 2 Estimation of the spectral radius of the iteration matrix for different iterative methods in the case of a FEM

Method	Number of iterations		
	40	80	120
SOR $w = 1.9$	0.98976	0.98021	0.97512
SEIDEL	0.98675	0.970929	0.96099
Jacobi	0.99213	0.98513	0.97928
JA	0.96728	0.96649	0.96412
SORA	0.96702	0.96223	0.96217

Table 3 Comparison of number of iterations necessary for convergence ($\mu = 0.0001$) in the case of HFEM and FEM

Method	Number of iterations	
	FEM	HFEM
SOR $w = 1.9$	179	121
SEIDEL	186	113
Jacobi	201	126
JA	92	42
SORA	77	31

Figure 1 Dependence of $R^{(n)}$ on the number of iterations (SORA w = 1.662). A, Solution with 8 hierarchical functions; B, solution with non-hierarchical functions; C, solution with 16 hierarchical functions

Figure 1 illustrates the efficiency of the HFEM. It shows, for the case of the SORA algorithm the convergence of $R^{(n)}$ towards a stationary value, theoretically equal to λ_1 the spectral radius of the iteration matrix. It is readily seen that the rate of convergence is fastest and monotone when hierarchical basis functions are introduced (graphs A and C). Convergence is towards the value $\lambda_1 = 0.6946$. When non-hierarchical basic functions are used, the rate of convergence is slower and convergence is towards the higher value of $\lambda_1 = 0.864$ (graph B). Observe also that the introduction of new hierarchical basic functions does not affect the rate of convergence after a certain threshold is reached. This is clear from graph A (8 functions) and graph C (16 functions) which are practically the same.

Figures 2 and *3* show, for two distinct values of θ ($\theta_1 = 3\pi/2, \theta_2 = \pi/2$) the distribution in the $x-y$ geometry of the neutron flux obtained from the solution of the test problem. As the source is anisotropic, the flux distribution is directly linked to the angle. The agreement between these solutions and expected results (from the physics of the situation) show that the HFEM is an efficient method for solving the neutron transport problem.

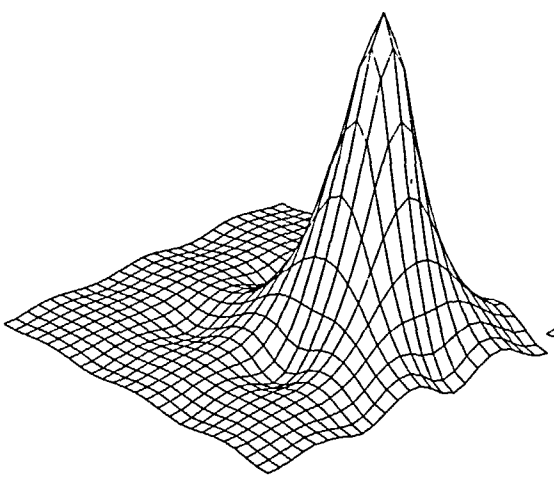

Figure 2 Distribution of the angular flux ($\theta = \pi/2$)

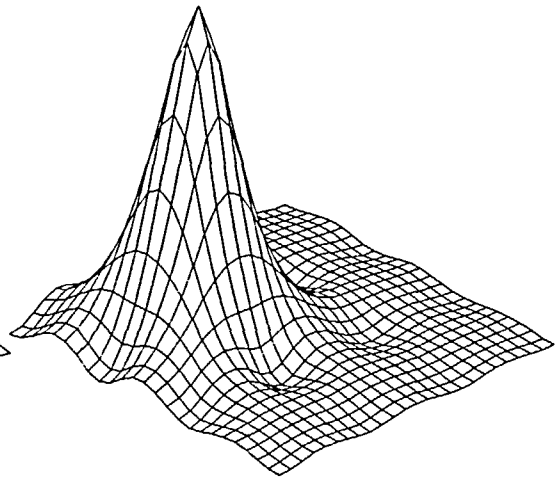

Figure 3 Distribution of the angular flux ($\theta = 3\pi/2$)

CONCLUSION

This paper has shown that the application of the HFEM to the neutron transport equation leads to block-systems of linear equations with desirable properties: positive definiteness, symmetry, diagonal dominance. These properties are conducive to the application of iterative methods for the solution of the HFEM mesh equations.

Numerical examples presented show the efficiency of the adaptive SOR method compared to other iterative algorithms (JACOBI, JA, SOR, SEIDEL). Because of the reduction of the spectral radius of the iteration matrix in the case of a HFEM, the convergence to the solution by iterative methods is faster than in the case of a non-hierarchical basis. This feature of the method is particularly useful in solving the neutron transport equation.

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